Magnetic circuits

- Motivation: Magnetic circuits provide the theory behind transformers. It will also provide the basics to study electromechanical energy conversion later in the course.
- Key ideas to learn:
  1. current produces magnetic field.
  2. changing flux drives a voltage.

We can utilize these together to make inductors & transformers.

- Key concepts that will help us:
  - Ampere's law
  - Gauss' law
  - Faraday's law

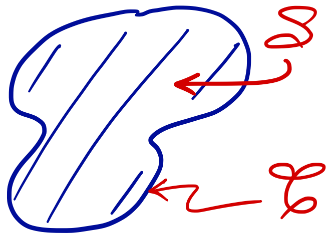
  } part of Maxwell's eqⁿ's.

Let's begin our study from Ampere's law.

$$\oint_{\mathcal{C}} \vec{H} \cdot d\vec{l} = \int_{\mathcal{S}} \vec{J}_f \cdot \hat{n} \, da$$

$\mathcal{S}$: open surface

$\mathcal{C}$: boundary of $\mathcal{S}$.

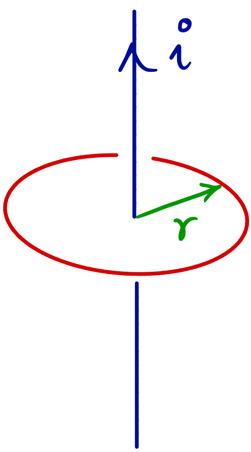


$\vec{H}$: Magnetic field intensity.

$\vec{J}_f$: Current density

$d\vec{l}$: Differential length element along $\mathcal{C}$.

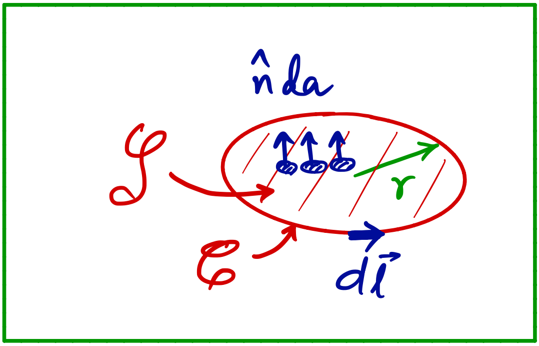
$(\hat{n} \, da)$: Differential area element on $\mathcal{S}$.



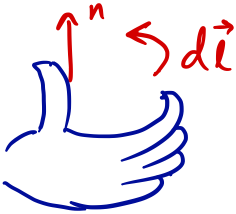
- Consider an infinitely long conductor carrying current i. Compute $\vec{H}$ induced. By symmetry, $\vec{H}$ is tangential and $|\vec{H}|$ only depends on r.

Ampere's law $\Rightarrow$ $\displaystyle\int_{\mathcal{C}} \vec{H}\cdot d\vec{l} = \int_{\mathcal{S}} \bar{J}_f \cdot \hat{n}\, da$

Geometry of $\mathcal{S}$ and $\mathcal{C}$ in applying Ampere's law.



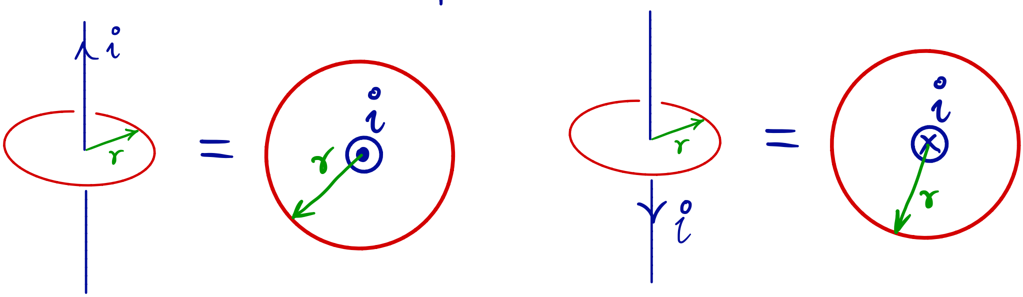
Direction of $\hat{n}$ depends on $d\vec{l}$. It is given by the right-hand rule.

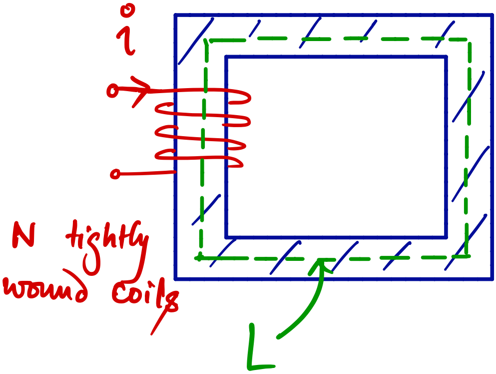


Back to our example:

$$|\vec{H}(r)|\cdot 2\pi r = i \Rightarrow |\vec{H}(r)| = \frac{i}{2\pi r}.$$

Notation: View from the top.





• Consider an iron piece with a cross sectional area A, and an average length around the loop being L as shown.

Assume that $\vec{H}$ has uniform magnitude throughout the iron piece, compute its magnitude.

Ampere's law yields $H \cdot L = N \cdot i$

$$\Rightarrow H = \frac{Ni}{L}.$$

Unit of H? Ampere-turns/meter.

The magnetic field $\vec{H}$ magnetizes the iron core. Assume that it is a "linear magnetic material".

Linear magnetic material

$\Rightarrow \vec{B} \propto \vec{H}$,

where $\vec{B}$ = magnetic flux density.

$\vec{B} \propto \vec{H}$. The proportionality constant depends on the material in which the magnetic field exists. It is called the permeability of the material, denoted by μ.

$$\therefore \vec{B} = \mu \vec{H}.$$

In our example, $H = \dfrac{Ni}{L}$

$$\Rightarrow B = \frac{\mu N}{L} \cdot i.$$

Magnetic flux through the material is given by $\displaystyle\int_{\text{cross-sectional area}} \vec{B} \cdot d\vec{s}$, denoted by ϕ.

In our example, $\phi = B \cdot A$

$$= \frac{\mu N i}{L} A.$$

- Units : ϕ is measured in Webers (Wb).
  $\vec{B}$ " " " Wb/m², also called Tesla.

- Let's look at this relation a little more closely.

$$\phi = \mu \frac{N i A}{L} = \frac{N i}{L/\mu A}.$$

This equation looks like

$$\text{current}\,(\phi) = \frac{\text{EMF}\,(Ni)}{\text{resistance}\,(L/\mu A)}.$$

Due to this resemblence, we call

Ni as magneto-motive force (mmf) $\mathcal{F}$,

$\frac{L}{\mu A}$ as reluctance $\mathcal{R}$.

$$\therefore \quad \phi = \frac{\mathcal{F}}{\mathcal{R}}.$$

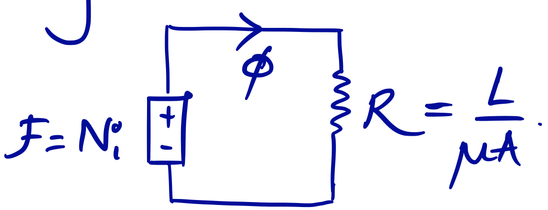
$1/\mathcal{R}$ is often called permeance.

The more magnetizeable a material is, the larger its μ is.

$\Rightarrow$ the smaller its reluctance $\mathcal{R} = \frac{L}{\mu A}$.

$\Rightarrow$ the larger the flux $\phi = \mathcal{F}/\mathcal{R}$ that flows through the material.

- The equation $\phi = \mathcal{F}/\mathcal{R}$ suggests that one can draw a hypothetical magnetic circuit



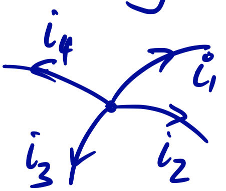
- Does flux satisfy a Kirchhof's current law type relation?

Yes, indeed! This is given by Gauss' law that states

$$\oint_S \vec{B}\cdot d\vec{s} = 0$$

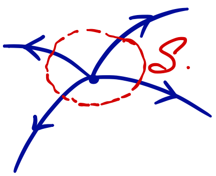
a closed surface

KCL says summation of currents leaving a pt. in the circuit $= 0$

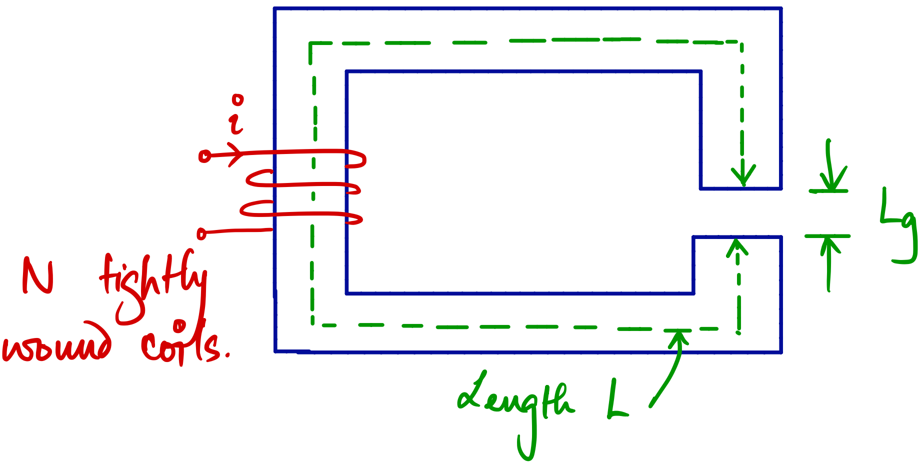


$$i_1 + i_2 + i_3 + i_4 = 0.$$

Let's write this relation in integral form.



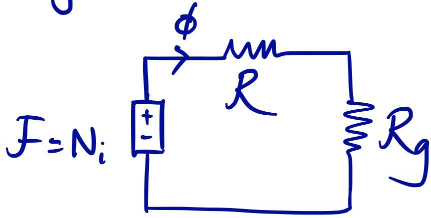
$$\oint_S \vec{J}\cdot d\vec{s} = 0$$ .... Similar to Gauss' law.



- Consider the iron piece of uniform cross sectional area A. Assume that the air gap has length g. The iron has a permeability of μ and air has permeability of μ_0. Calculate the magnetic flux through the core.

Magnetic circuit:

$$R = \frac{L}{\mu A}, \quad R_g = \frac{L_g}{\mu_0 A}.$$



$$\phi = \frac{\mathcal{F}}{R + R_g} = \frac{Ni}{\frac{L}{\mu A} + \frac{L_g}{\mu_0 A}}.$$

- The magnetic circuit is not a circuit through which current flows. It is an abstraction that allows us to circumvent the use of Ampere's law & Gauss law, by using Ohm's law type relations.

V. Imp!

- $\mu_0 = 4\pi \times 10^{-7}$ SI units.

  Units of μ_0 = Tesla-meter/Ampere,

  = Wb/Amp-meter

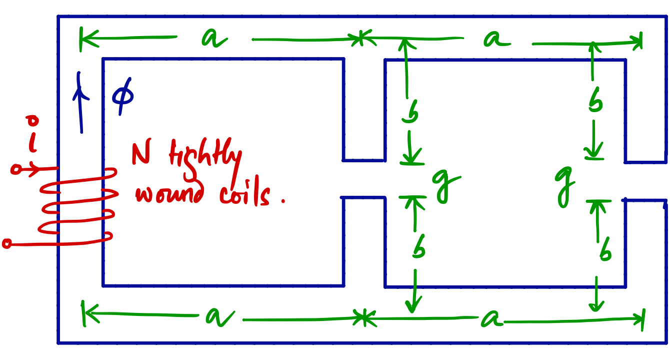
$\mu_{Iron} = 6.3 \times 10^{-3}$ Wb/Am.

$\therefore \frac{\mu_{Iron}}{\mu_0}$ (called the "relative permeability") $= 5000$.

Often transformer cores use cobalt-iron, that has a relative permeability $\mu_r \approx 18{,}000$.

$\Rightarrow R_{air-gap} \gg R_{iron}$.

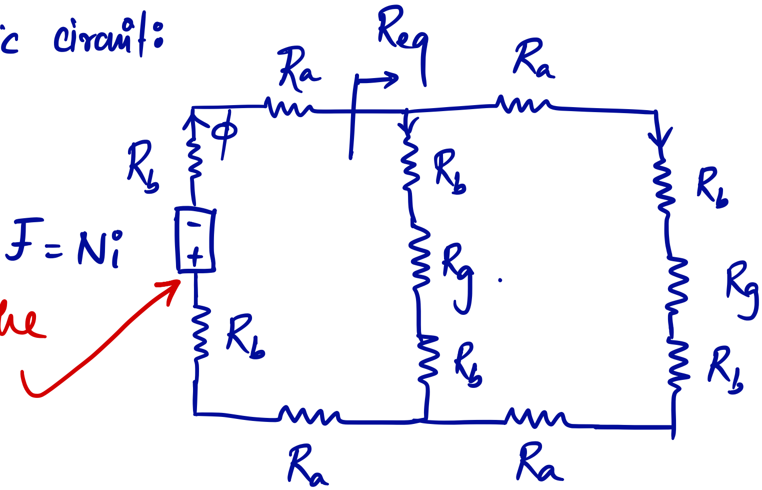
• Another example



Assume uniform cross-section A of the iron piece with permeability μ. Compute ϕ as shown.

Also, assume $b + g/2 \approx b$.

Magnetic circuit:



Notice the polarity

$$R_a = \frac{a}{\mu A}, \quad R_b = \frac{b}{\mu A}, \quad R_g = \frac{g}{\mu_0 A}, \quad R_{2b+g} = \frac{2b+g}{\mu A}$$

To compute ϕ, let's club the reluctances.

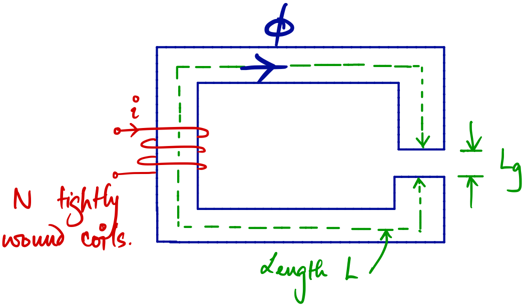
$$R_{eq} = (2R_b + R_g) \parallel (2R_a + 2R_b + R_g).$$

$$\therefore \phi = \frac{-F}{2R_a + 2R_b + (2R_b + R_g) \parallel (2R_a + 2R_b + R_g)}$$

The notation $R_1 \parallel R_2 := \dfrac{R_1 R_2}{R_1 + R_2}$.

- <u>Defining self-inductance:</u>

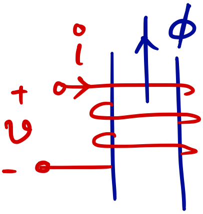
Let's get back to the simplest example



Recall that we computed ϕ.

$$\phi = \frac{Ni}{\frac{L}{\mu A} + \frac{L_g}{\mu_0 A}}.$$

Notice that if the current changes, so will the flux, and hence, the flux linkage with the coil.



Flux linked with the coil

$$\lambda = N\phi .$$

Voltage induced $v = \dfrac{d\lambda}{dt}$.

$$\Rightarrow \quad v = \frac{d}{dt}(N\phi)$$

$$= N \cdot \frac{d}{dt}\left[\frac{Ni}{\frac{L}{\mu A} + \frac{L_g}{\mu_0 A}}\right]$$

$$= \underbrace{\frac{N^2}{\frac{L}{\mu A} + \frac{L_g}{\mu_0 A}}}_{:= \mathcal{L}} \cdot \frac{di}{dt}$$

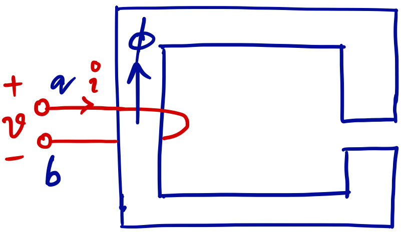
$$= \mathcal{L} \cdot \frac{di}{dt} .$$

$\mathcal{L}$ is called the self-inductance or inductance of the coil.

• Where does $v = \frac{d\lambda}{dt}$ come from?

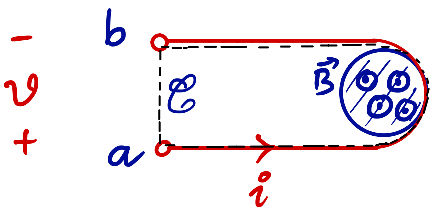
Faraday's law: $\oint_{\mathscr{C}} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \oint_{\mathscr{S}} \vec{B} \cdot d\vec{s}$

$\mathscr{S}$: open surface, $\mathscr{C}$: contour of $\mathscr{S}$



To simplify, let's consider one loop.

Let's look at this from the top.



Black dotted line defines $\mathscr{C}$ that encloses the surface $\mathscr{S}$.

$\oint_{\mathscr{C}} \vec{E} \cdot d\vec{l} = -v$.

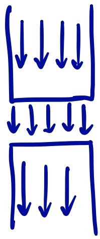
and $\oint_{\mathscr{S}} \vec{B} \cdot d\vec{s} = \phi$.

$\Rightarrow v = \frac{d\phi}{dt}$. For N turns, $v = N \cdot \frac{d\phi}{dt} = \frac{d\lambda}{dt}$.

• In our derivation, we have used

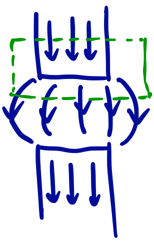
$$R_{iron} = \frac{L}{\mu A}, \quad R_{air\text{-}gap} = \frac{L_g}{\mu_0 A}.$$

why is this A ?



Assumption: The uniform flux path inside the conductor also induces uniform flux through the air-gap.

$$\therefore R_{air\text{-}gap} = \frac{L_g}{\mu_0 A}.$$



Reality: Total flux inside the iron core $= B_{iron} \cdot A_{iron}$

Flux in air-gap $= B_{air\text{-}gap} \cdot A_{airgap}$.

$A_{air\text{-}gap} > A_{iron}$ ... due to "fringing" of the flux-paths.

Sometimes, one uses $A_{airgap} = 1.1\, A_{iron}$.